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GPS positioning and Dilution of Precision

Laboratory assignment 2

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1. Aim of the study:

The purpose of this laboratory practice is to evaluate the Dilution of Precision; **DoP** by using previous developed code, in fact it will be modified as there is no need to plot the constellations. The code will be complemented with new functions that will permit the assessment of the GPS positioning accuracy.

The main factors regarding the precision are:

a) Errors related with relativistic effects and interaction with the ionosphere which causes variation in the signals received by the user, these have an impact on the pseudo ranges which indeed provides erroneous position determination, they can be mitigated to some effect by using models of those errors in the receiver.

b) The location of the satellites regarding the receiver: while solving the equations for the position bigger errors than the ones of the pseudo ranges could occur, this requires the implementation of a matrix with a condition number.

Finally, the last factors are a consequence of the first ones, which are only amplified. This leads to the DoP, which can be accounted in advance if we are expecting to determine a certain position by taking measurements when the DoP is minimum.

Satellite coverage:

In the first practice we computed the ECEF coordinates of the GPS satellites from their Keplerian elements, also by using the TLE system.

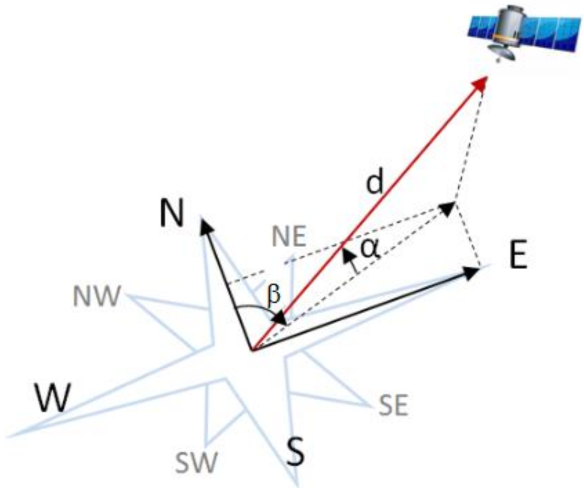
With that information and knowing the user's position we can determine the line of sight of the satellites visible of the constellation:

```
% EETAC  Coordinates Google Earth
lat = 41.2755182;
lon = 1.9866707;
alt = 5;

% Visibility limitation for LoS
min_elev=10;
max_elev=90;
```

2. Azimuth, elevation and distance:

In order to determine if a satellite is visible, we need to obtain the ambles and distance between the satellite and the point of interest. The azimuth (β) is the horizontal angle to the satellite measured clockwise from a north base line and the elevation (α) is the angle between the horizon and the centre of the satellite.



Converting LLA coordinates to ECEF Cartesian

From the first Lab deliverable we can transform the ECEF coordinates to LLA, but we also need to convert the LLA coordinates of the EETAC to ECEF in order to be able to compute the LoS and the pseudoranges.

$$x = \left(\frac{a}{\sqrt{1 - e^2 \operatorname{sen}^2 \phi}} + h \right) \cos\phi \cdot \cos\lambda$$

$$y = \left(\frac{a}{\sqrt{1 - e^2 sen^2 \phi}} + h \right) \cos\phi \cdot sen\lambda$$

$$z = \left(\frac{a\left(1 - e^2\right)}{\sqrt{1 - e^2 sen^2 \phi}} + h \right) sen\phi$$

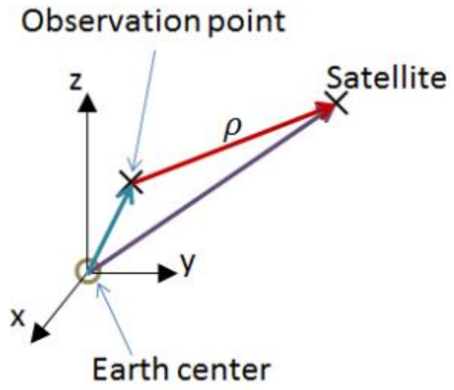
where λ is the geodetic longitude, ϕ is the geodetic latitude, h is the height above the ellipsoid (with a value of 4 metres) and a and e are the ellipsoid parameters taken as:

a = 6378137 m and $e^2 =$ 0.00669437999014

Computing vector $\boldsymbol{\rho}$

Computing the vector ρ is achieved by subtracting the vectors of the user position (observation point) and the satellite ECEF coordinates:

$$\bar{\rho} = \overline{Sat} - \overline{Obs}$$



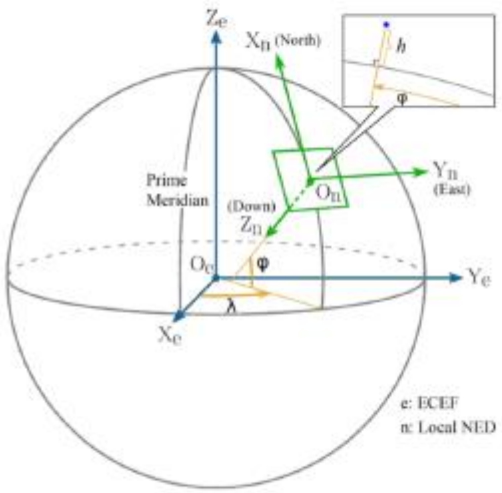
The coordinates obtained are given in a system parallel to ECEF centred at EETAC's position. Thus, we must rotate the axes to transform the vector to NED (North, East, Down)

Rotating axes from Cartesian ECEF to Cartesian NED coordinates

North-East -Down **NED** is the coordinate system used for navigation. It is fixed to the Earth's surface and positioned over the ellipsoid WGS-84 model.

The main characteristics for this model are:

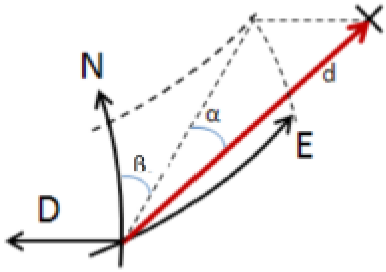
- The origin (denoted by On) is arbitrarily fixed to a point on the earth's surface.
- The X-axis (denoted by Xn) points toward the ellipsoid north (geodetic north).
- The Y-axis (denoted by Yn) points toward the ellipsoid east (geodetic east).
- The Z-axis (denoted by Zn) points downward along the ellipsoid normal.



To obtain the NED coordinates it is as simple as to multiply by its corresponding matrix, which takes into account the latitude and longitude of the observation point:

$$\begin{bmatrix} N \\ E \\ D \end{bmatrix} = \begin{bmatrix} -sin\emptyset\, cos\lambda & -sin\emptyset\, sin\lambda & cos\emptyset \\ -sin\lambda & cos\lambda & 0 \\ -cos\emptyset\, cos\lambda & -cos\emptyset\, sin\lambda & -sin\emptyset \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now we are in position to obtain the elevation and azimuth angles:



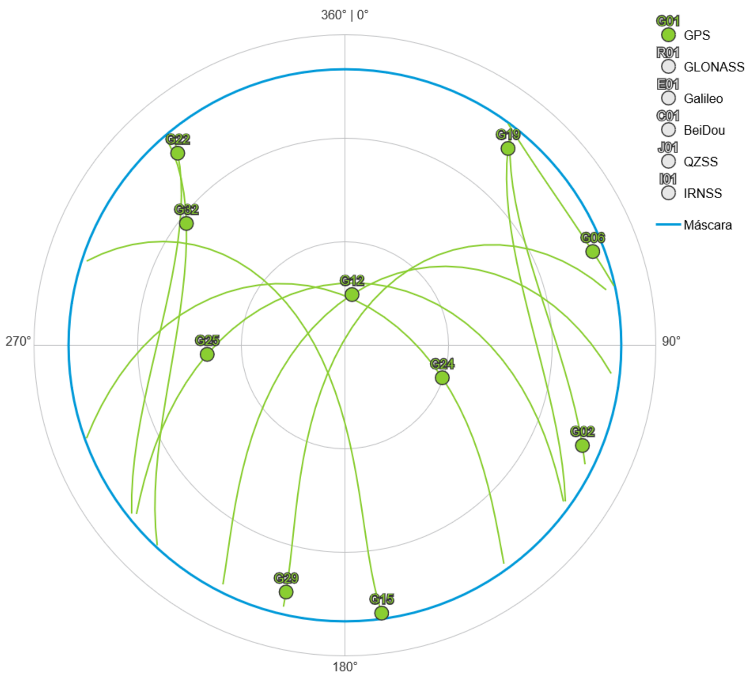
$$d = \sqrt{N^2 + E^2 + D^2}$$

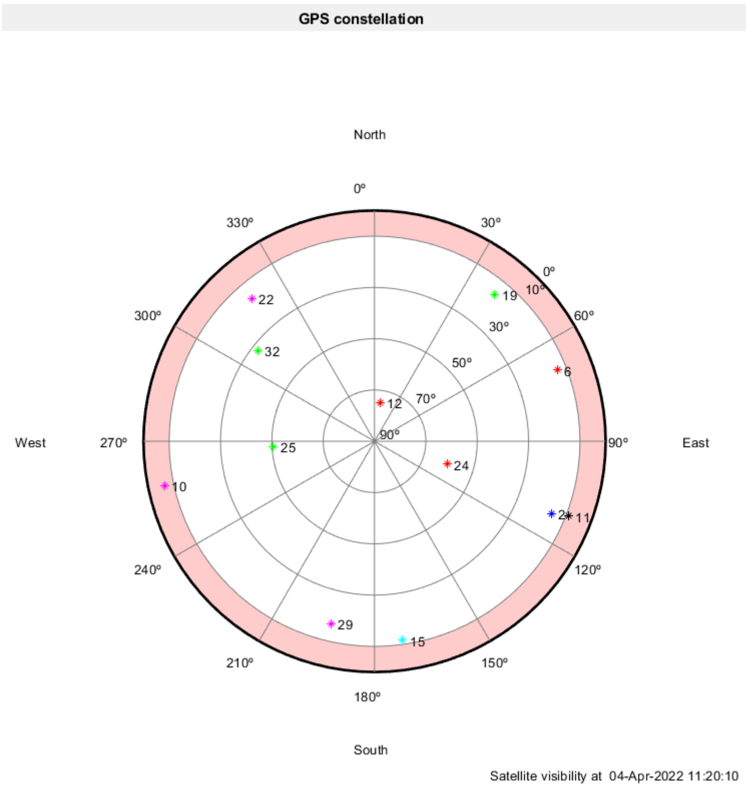
$$\alpha = \operatorname{asin}\left(-\frac{D}{d}\right)$$

$$\beta = \operatorname{atan}\left(\frac{E}{N}\right)$$

With all the angles obtained, we cand discard the satellites that are not visible from our observation point (EETAC)

Comparing obtained from MATLAB with http://www.gnssplanningonline.com/#/skyplot:





3. The GPS navigation system of equations:

The pseudo ranges can be obtained as follows:

$$\rho_j = \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} - c \cdot t_u = f(x_u, y_u, z_u, t_u)$$

Where subindex *j* stands for a certain satellite and *u* refers to the user. It is important to mention that we are working with the ECEF system.

"Since this expression contains 4 unknowns, we need at least 4 independent equations to solve the system. Therefore, a minimum of 4 pseudo range to 4 different satellites must be measured to find the position and the clock error (notice that the time error is common to the 4 equations). The pseudo range is measured at the receiver by computing the correlation between the received PRN sequences and their local version. It's called a pseudo range because it depends on the range (distance) to the satellite but also on the time error and other errors (mainly the ionosphere error, since the pseudo range assumes that the signal travels at the speed of light, which is not true inside the ionosphere)." – NAAC practice 2 guide.

This section will be explained using the SoW guide which I give the credit for:

Although the pseudo range equations are not linear, the equation system can be solved using a linearization method. In order to obtain a linear system, the pseudo range ρj is expressed as:

$$\rho_j = f(x_u, y_u, z_u, t_u) = f(\hat{x}_u + \Delta x_u, \hat{y}_u + \Delta y_u, \hat{z}_u + \Delta z_u, \hat{t}_u + \Delta t_u)$$

where $\hat{x}_u$ $\hat{y}_u$ $\hat{z}_u$ and $\hat{t}_u$ are the user estimated position and time (known) and $\Delta\hat{x}_u$, $\Delta\,\hat{y}_u$, $\Delta\hat{z}_u$ and $\Delta\hat{t}_u$ are the increments (unknown) with respect to the estimated values.

By expressing the function as a Taylor series (centred at the estimated values) and keeping only the linear terms we get the approximated expression:

$$\rho_j \cong f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u) + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{x}_u}\Delta x_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{y}_u}\Delta y_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{z}_u}\Delta z_u + \frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{t}_u}\Delta t_u$$

Where all the partial derivatives can be computed to give:

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{x}_u} = -\frac{x_j - \hat{x}_u}{\hat{r}_j} \equiv -a_{xj}$$
$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{y}_u} = -\frac{y_j - \hat{y}_u}{\hat{r}_j} \equiv -a_{yj}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{z}_u} = -\frac{z_j - \hat{z}_u}{\hat{r}_j} \equiv -a_{zj}$$

$$\frac{\partial f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)}{\partial \hat{t}_u} = -c$$

Where $\hat{r}_j$ is the estimated range to the satellite (which can be computed based on the user estimated position)

$$\hat{r}_j = \sqrt{\left(x_j - \hat{x}_u\right)^2 + \left(y_j - \hat{y}_u\right)^2 + \left(z_j - \hat{z}_u\right)^2}$$

Notice that estimated pseudorange is <u>not measured</u>, but computed; so it contains no errors. Based on $\hat{\rho}_j$ we can write the approximated pseudorange as:

$$\rho_j \cong \hat{\rho}_j - a_{xj} \cdot \Delta\mathrm{x_u} - a_{yj} \cdot \Delta\mathrm{y_u} - a_{zj} \cdot \Delta\mathrm{z_u} - \mathrm{c} \cdot \Delta\mathrm{t_u}$$

And defining $\Delta\rho_j = \hat{\rho}_j - \rho_j$ we get:

$$\Delta\rho_j = a_{xj} \cdot \Delta\mathrm{x_u} + a_{yj} \cdot \Delta\mathrm{y_u} + a_{zj} \cdot \Delta\mathrm{z_u} + \mathrm{c} \cdot \Delta\mathrm{t_u}$$

In case of measuring the pseudorange to 4 different satellites we get the following equation system:

$$\begin{aligned}
\Delta\rho_1 &= a_{x1} \cdot \Delta\mathrm{x_u} + a_{y1} \cdot \Delta\mathrm{y_u} + a_{z1} \cdot \Delta\mathrm{z_u} + \mathrm{c} \cdot \Delta\mathrm{t_u} \\
\Delta\rho_2 &= a_{x2} \cdot \Delta\mathrm{x_u} + a_{y2} \cdot \Delta\mathrm{y_u} + a_{z2} \cdot \Delta\mathrm{z_u} + \mathrm{c} \cdot \Delta\mathrm{t_u} \\
\Delta\rho_3 &= a_{x3} \cdot \Delta\mathrm{x_u} + a_{y3} \cdot \Delta\mathrm{y_u} + a_{z3} \cdot \Delta\mathrm{z_u} + \mathrm{c} \cdot \Delta\mathrm{t_u} \\
\Delta\rho_4 &= a_{x4} \cdot \Delta\mathrm{x_u} + a_{y4} \cdot \Delta\mathrm{y_u} + a_{z4} \cdot \Delta\mathrm{z_u} + \mathrm{c} \cdot \Delta\mathrm{t_u}
\end{aligned}$$

which is a linear system that can be also written in matrix notation:

$$\mathbf{\Delta\rho} = \mathbf{H} \cdot \mathbf{\Delta u}$$

In that system the known data are: the incremental pseudorange vector:

$$\mathbf{\Delta\rho} = \begin{bmatrix} \Delta\rho_1 \\ \Delta\rho_2 \\ \Delta\rho_3 \\ \Delta\rho_4 \end{bmatrix}$$

and the unitary vectors from the user estimated position (matrix $\mathbf{H}$):

$$\mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ a_{x3} & a_{y3} & a_{z3} & 1 \\ a_{x4} & a_{y4} & a_{z4} & 1 \end{bmatrix}$$

While the unknown vector is the user incremental position:

$$\mathbf{\Delta u} = \begin{bmatrix} \Delta x_u \\ \Delta y_u \\ \Delta z_u \\ c \cdot \Delta t_u \end{bmatrix}$$

When the number of available satellites is exactly 4 the system can be solved by inverting matrix H:

$$\mathbf{\Delta u} = \mathbf{H}^{-1} \cdot \mathbf{\Delta \rho}$$

However, when more than 4 satellites are available, which is the usual situation, the system becomes over specified (more equations than unknowns) and it has not an exact solution. In this case we must assume that the pseudo range vector contains errors and solve the system in the sense of the least square error. The least square error solution is based on the pseudo-inverse matrix and can also be applied when only 4 satellites are available:

$$\mathbf{\Delta u} = \left(\mathbf{H}^{T}\mathbf{H}\right)^{-1}\mathbf{H}^{T} \cdot \mathbf{\Delta \rho}$$

This solution must be computed repeatedly until the magnitude of Δu becomes sufficiently small (to ensure that the Taylor series approximation is realistic).

Computing matrix H:

However, in our case we have to compute matrix H using all visible satellites computed previously instead of solving the navigation system of equations. This matrix will have 4 columns and 31 rows (total number of GPS satellites in the constellation). The rows of H are simply the 3 Cartesian coordinates of the unitary vectors pointing from the user to each of the satellites

4. Dilution of precision:

The DoP is how the propagation of error conditionate the final computed position, the effect is caused by the processing of the pseudo ranges at the receiver where they are amplified as they propagate.

It only depends on the system matrix condition which is related with the positions of the satellites servicing the receiver. Therefore, the DoP can be predicted when we are looking for fine measurements by finding the optimal window.

As initially stated, there is an error while computing the pseudo range, that can be mitigated but never totally eliminated, taking it into account:

$$\mathbf{\Delta \rho}' = \mathbf{\Delta \rho} + \boldsymbol{\eta} = \mathbf{H} \cdot \mathbf{\Delta u} + \boldsymbol{\eta}$$

where $\mathbf{\Delta\rho'}$ is the measured pseudo range vector (with errors), $\mathbf{\Delta\rho}$ is the theoretical pseudo range vector (including only the time errors) and $\boldsymbol{\eta}$ is the pseudo range error vector.

The least squares solution of the previous equation is:

$$\mathbf{\Delta u'} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}} \cdot \mathbf{\Delta\rho'} = \mathbf{\Delta u} + \left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}} \cdot \boldsymbol{\eta} = \mathbf{\Delta u} + \boldsymbol{\varepsilon}$$

This means that we cannot obtain the user position without errors. The error vector affecting the obtained position is given by:

$$\boldsymbol{\varepsilon} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathrm{T}} \cdot \boldsymbol{\eta}$$

We can assume that all the components of $\boldsymbol{\eta}$ are independent zero

mean random variables with the same variance: σ^2.Ignoring for the moment the

matrix $(\mathrm{H}^{\mathrm{T}}\mathrm{H})^{-1}$ the vector that produces the positioning errors in terms of distance if

contributed by the multiplication of the first 3 rows of matrix $\mathbf{H}^{\mathbf{T}}$ by vector $\boldsymbol{\eta}$, that is:

$$\begin{bmatrix} a_{x1} \cdot \eta_1 + a_{x2} \cdot \eta_2 + a_{x3} \cdot \eta_3 + a_{x4} \cdot \eta_4 \\ a_{y1} \cdot \eta_1 + a_{y2} \cdot \eta_2 + a_{y3} \cdot \eta_3 + a_{y4} \cdot \eta_4 \\ a_{z1} \cdot \eta_1 + a_{z2} \cdot \eta_2 + a_{z3} \cdot \eta_3 + a_{z4} \cdot \eta_4 \end{bmatrix}$$

The squared magnitude of this error vector is:

$$(a_{x1} \cdot \eta_1 + a_{x2} \cdot \eta_2 + a_{x3} \cdot \eta_3 + a_{x4} \cdot \eta_4)^2 + \left(a_{y1} \cdot \eta_1 + a_{y2} \cdot \eta_2 + a_{y3} \cdot \eta_3 + a_{y4} \cdot \eta_4\right)^2 + (a_{z1} \cdot \eta_1 + a_{z2} \cdot \eta_2 + a_{z3} \cdot \eta_3 + a_{z4} \cdot \eta_4)^2$$

The average magnitude of this error vector is $4\sigma^2$ (if it is taken into account that $\boldsymbol{\eta}$ vector is formed by independent zero mean random variables). As the value is $4\sigma^2$ is proved that the multiplication of HT by $\boldsymbol{\eta}$ is not the source of positioning error because it is the same average error of the magnitude of $\boldsymbol{\eta}$.

The only possible cause of an error occurrence comes therefore from matrix Q=$(\mathrm{H}^{\mathrm{T}}\mathrm{H})^{-1}$

Assuming that all the components of $\boldsymbol{\eta}$ are independent zero mean random variables with the same variance (σ^2), only the diagonal elements of $\mathbf{Q}$ are relevant for the DOP calculation.

$$\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^{\mathrm{T}} = \mathbf{QH}^{\mathrm{T}} \cdot \boldsymbol{\eta}\boldsymbol{\eta}^{\mathrm{T}} \cdot \mathbf{HQ}^{\mathrm{T}} = \mathbf{QH}^{\mathrm{T}} \cdot \sigma^2\mathbf{I} \cdot \mathbf{HQ}^{\mathrm{T}} = \sigma^2 \cdot \mathbf{Q} \cdot \mathbf{H}^{\mathrm{T}}\mathbf{H} \cdot \mathbf{Q}^{\mathrm{T}} = \sigma^2\mathbf{Q}^{\mathrm{T}} = \sigma^2\mathbf{Q}$$

$$\mathbf{Q} = \begin{bmatrix} Q_{xx} & Q_{yx} & Q_{zx} & Q_{ctx} \\ Q_{xy} & Q_{yy} & Q_{zy} & Q_{cty} \\ Q_{xz} & Q_{yz} & Q_{zz} & Q_{ctz} \\ Q_{xct} & Q_{yct} & Q_{zct} & Q_{ctct} \end{bmatrix}$$

σx, σy and σz are the error standard deviations in ECEF coordinates, therefore they will be needed to be multiplied by the rotation matrix in order to obtain NED: σn, σE and σD.

Obtaining the following matrix:

$$\mathbf{Q'}=\mathbf{R}\cdot\mathbf{Q}\cdot\mathbf{R}^{\mathrm{T}}$$

$$\mathbf{Q'} = \begin{bmatrix} q_{NN} & q_{NE} & q_{ND} & q_{Nct} \\ q_{EN} & q_{EE} & q_{ED} & q_{Ect} \\ q_{DN} & q_{DE} & q_{DD} & q_{Dct} \\ q_{ctN} & q_{ctE} & q_{ctD} & q_{ctct} \end{bmatrix}$$

The Horizontal DOP (HDOP) is the multiplication factor that increases the distance errors in the horizontal plane. It is given by:

$$HDOP = \sqrt{q_{NN} + q_{EE}}$$

The Vertical DOP computes the error when computing the altitude:

$$VDOP = \sqrt{q_{DD}}$$

The Position DOP (PDOP) is the multiplication factor that increases the 3D distance:

$$PDOP = \sqrt{Q_{xx} + Q_{yy} + Q_{zz}} = \sqrt{q_{NN} + q_{EE} + q_{DD}} = \sqrt{HDOP^2 + VDOP^2}$$

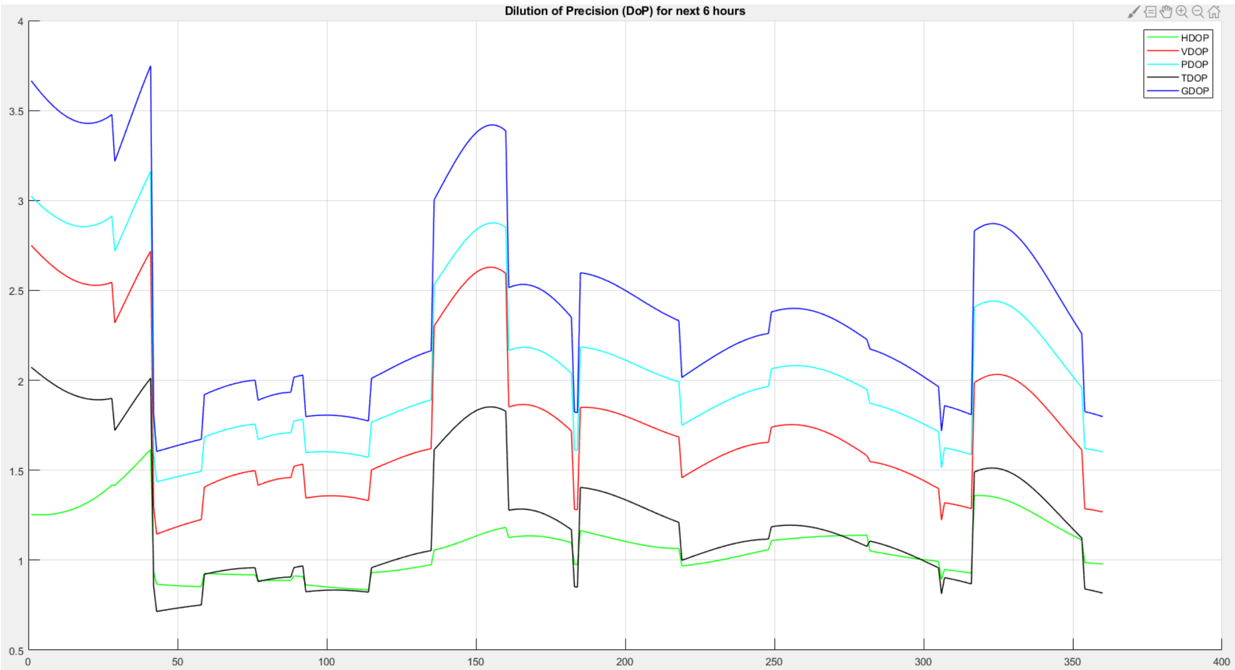
The Time DOP (TDOP) is the multiplication factor that increases the time error:

$$TDOP = \sqrt{q_{ctct}}$$

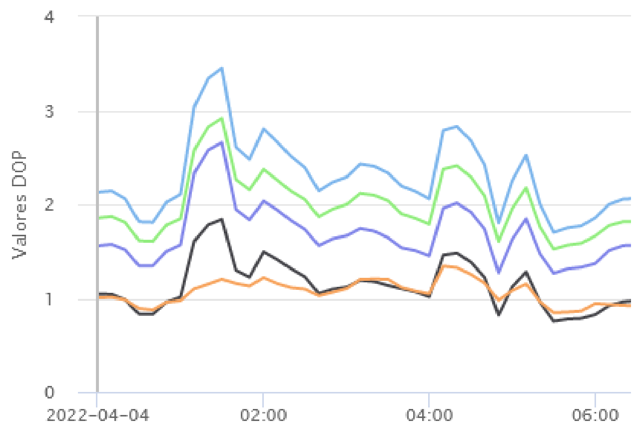
The Geometric DOP (GDOP) takes into account the position and time error increase factor:

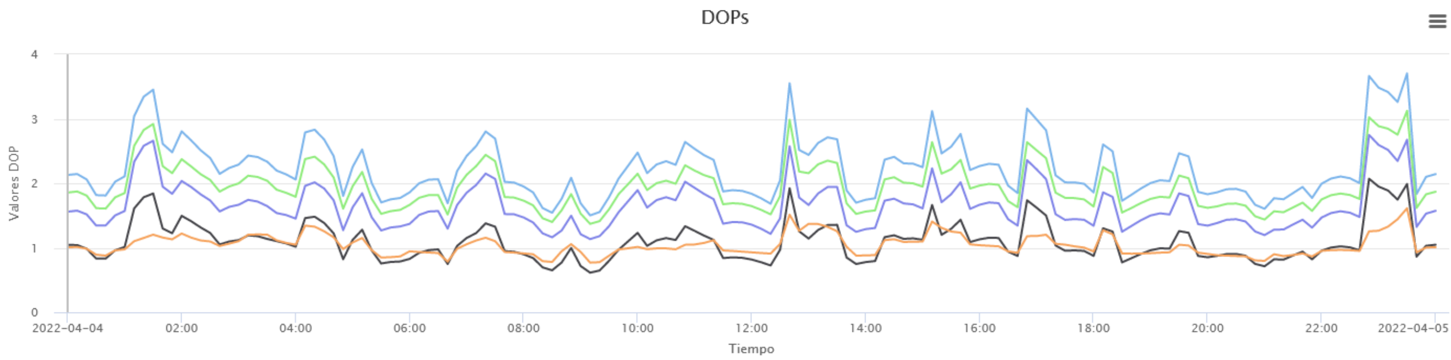
$$GDOP = \sqrt{PDOP^2 + TDOP^2}$$

The final result applying what has been stated in MATLAB is the following



Comparing the results with http://www.gnssplanningonline.com/#/charts we can see a similar shape, validating the obtained results.





5. Conclusions:

From the results, the most optimal window to do measurements would at 05:00 and at 00:40.

In this practice I learned how to determine the visible satellites in a specific position as well as the computation of DoP and its understanding, the first part was simple, even though I had to change my code regarding practice one in order to compute the ECEF coordinates as it was more orientated to plotting the position of the satellites.

The second part was the most complicated one, luckily, I could follow the instructions given in the SoW which are also a pilar when explaining all the theory in this report.

6. Bibliography:

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6. ***NACC SoW***